数理解析研究所講究録
963巻 1996年 55-62

INCLUSION PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

LI JIAN LIN (西北工業大学)
SHIGEYOSHI OWA (近畿大学・理工)

ABSTRACT. The object of the present paper is to give sharp forms of inclusion properties of the class $P(p,\alpha,\beta)$ under operators $J_{p,c}$, F_m and $J_{p,1}^{\lambda}$.

1. INTRODUCTION

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be a member of the class $P(p,\alpha)$ if it satisfies the inequality $\text{Re}\{f'(z)/z^{p-1}\} > \alpha$ $(z \in U)$ for some α $(0 \leqq \alpha < p)$. The classes $P(1,0)$ and $P(p,0)$ were investigated by MacGregor [5] and Umezawa [6], respectively.

For a function $f(z)$ belonging to $A(p)$, we define the generalized Bernardi integral operator $J_{p,c}$ by

$$J_{p,c}(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt$$

$$= z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} a_{p+n} z^{p+n} \qquad (p \in N;\ c > -p). \tag{1.2}$$

The operator $J_{1,c}$ for $c \in N$ was introduced by Bernardi [1]. Clearly, from (1.2) we see that

$$f(z) \in A(p) \implies J_{p,c}(f) \in A(p) \qquad (c > -p). \tag{1.3}$$

1991 Mathematics Subject Classification. Primary 30C45.

Thus, by applying the operator $J_{p,c}$ successively, we can obtain

$$J^n_{p,c}(f) = \begin{cases} J_{p,c}(J^{n-1}_{p,c}(f)) & (n \in \mathbb{N}) \\ f(z) & (n = 0). \end{cases} \tag{1.4}$$

Suppose also that

$$\begin{aligned} F_m(f) &= J_{p,c_m}(J_{p,c_{m-1}} \cdots (J_{p,c_1}(f))) \\ &= z^p + \sum_{n=1}^{\infty} \left[\prod_{j=1}^{m} \frac{p+c_j}{p+n+c_j} \right] a_{p+n} z^{p+n} \qquad (c_j > -p;\ m \in \mathbb{N}). \end{aligned} \tag{1.5}$$

For an analytic function $g(z)$ given by $g(z) = \sum_{n=0}^{\infty} b_{p+n} z^{p+n}$ in $\mathbb{U}$, and for a real number λ, Flett [3] define the multiplier transformation $I^{\lambda}g(z)$ by

$$I^{\lambda}g(z) = \sum_{n=0}^{\infty} (p+n+1)^{-\lambda} b_{p+n} z^{p+n} \qquad (z \in \mathbb{U}). \tag{1.6}$$

The function $I^{\lambda}g(z)$ is clearly analytic in $\mathbb{U}$. It may be regarded as a fractional integral (for $\lambda > 0$) or a fractional derivative (for $\lambda < 0$) of $g(z)$. Furthermore, in terms of the Gamma function, we have

$$I^{\lambda}g(z) = \frac{1}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} g(tz)\,dt \qquad (\lambda > 0). \tag{1.7}$$

Denote by $D^{\lambda}g(z)$ the multiplier transfomation $I^{-\lambda}g(z)$ for $\lambda \geqq 0$, i.e.,

$$D^{\lambda}g(z) = I^{-\lambda}g(z) = \sum_{n=0}^{\infty} (p+n+1)^{\lambda} b_{p+n} z^{p+n} \qquad (\lambda \geqq 0;\ z \in \mathbb{U}). \tag{1.8}$$

From (1.4) and (1.6),

$$J^m_{p,1}(f) = (p+1)^m I^m(f) \qquad (m \in \mathbb{N};\ f \in A(p)). \tag{1.9}$$

Thus, one can define the operator $J^{\lambda}_{p,1}$ (depending on a continuous parameter $\lambda > 0$) by

$$J^{\lambda}_{p,1}(f) = (p+1)^{\lambda} I^{\lambda}(f) \qquad (\lambda > 0;\ f \in A(p)). \tag{1.10}$$

Making use of the fractional derivative operator and operators $J_{p,c}$, F_m and $J^{\lambda}_{p,1}$ as mentioned above, Cho [2] introduced and studied the class $P(p,\alpha,\beta)$ defined by

$$P(p,\alpha,\beta) = \{f \in A(p) : (p+1)^{-\beta}D^{\beta}f \in P(p,\alpha)\},$$

where $0 \leq \alpha < p$ and $\beta \geq 0$. Observe that $P(p,\alpha,0) = P(p,\alpha)$. If $\beta \geq 0$ and $0 \leq \alpha_1 \leq \alpha_2 < p$, then $P(p,\alpha_2,\beta) \subset P(p,\alpha_1,\beta)$. The class $P(1,\alpha,\beta)$ was introduced and studied by Kim, Lee and Srivastava [4]. In [2], Cho showed that

(i) if $f(z) \in P(p,\alpha,\beta)$, then $J_{p,c}(f)$, $F_m(f)$ and $J^{\lambda}_{p,1}(f)$ are slao in the class $P(p,\alpha,\beta)$, where $c \in N$ and $c_j \in N$,

(ii) if $0 \leq \alpha < p$ and $\beta \geq 0$, then $P(p,\alpha,\beta+1) \subset P(p,\mu,\beta)$, where $\mu = (2\alpha(p+1)+p)/(2(p+1)+1)$.

In the case of $p = 1$, these results correspond to the results by Kim, Lee and Srivastava [4]. In the present paper, we give the sharp forms of these results simply.

2. Inclusion Properties

Our first result for the class $P(p,\alpha,\beta)$ is contained in

THEOREM I. If $f(z)$ is in the class $P(p,\alpha,\beta)$, then $J_{p,c}(f)$ belongs to the class $P(p,\mu,\beta)$, where

$$\mu = p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c} .$$

The result is sharp.

PROOF. It follows from the definitions (1.2) and (1.8) that

$$(p+1)^{-\beta}D^{\beta}(J_{p,c}(f)) = J_{p,c}((p+1)^{-\beta}D^{\beta}(f))$$

$$= (p+c) \int_0^1 t^{c-1}\{(p+1)^{-\beta}D^{\beta}f(tz)\}dt. \tag{2.1}$$

Therefore, setting

$$H(z) = (p+1)^{-\beta}D^{\beta}(J_{p,c}(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta}D^{\beta}f(z), \tag{2.2}$$

we must show that

$$\operatorname{Re}\left(\frac{H'(z)}{z^{p-1}}\right) > \mu \qquad (0 \leqq \alpha < p;\ c > -p;\ z \in \mathcal{U}) \tag{2.3}$$

whenever $h(z) \in \mathcal{P}(p,\alpha)$. Note that (2.1) gives

$$\operatorname{Re}\left(\frac{H'(z)}{z^{p-1}}\right) = (p+c)\int_0^1 t^{p+c-1}\operatorname{Re}\left(\frac{h'(tz)}{(tz)^{p-1}}\right)dt. \tag{2.4}$$

Since $h(z) \in \mathcal{P}(p,\alpha)$, we have

$$\operatorname{Re}\left(\frac{h'(tz)}{(zt)^{p-1}}\right) > \frac{p-(p-2\alpha)t}{1+t} \qquad (0 < t \leqq 1;\ z \in \mathcal{U}) \tag{2.5}$$

and hence (2.4) yields

$$\operatorname{Re}\left(\frac{H'(z)}{z^{p-1}}\right) > (p+c)\int_0^1 t^{p+c-1}\,\frac{p-(p-2\alpha)t}{1+t}\,dt$$

$$= p + 2(p-\alpha)(p+c)\sum_{n=1}^{\infty}\frac{(-1)^n}{p+n+c}\,. \tag{2.6}$$

Further, to show that the result is sharp, we consider the function

$$f_0(z) = z^p + \sum_{n=1}^{\infty}\frac{2(p-\alpha)(p+1)^{\beta}}{(p+n)(p+n+1)^{\beta}}\,(-1)^n z^{p+n}\,, \tag{2.7}$$

which belongs to the class $\mathcal{P}(p,\alpha,\beta)$. Since

$$H_0(z) = (p+1)^{\beta}D^{\beta}(J_{p,c}(f_0(z)))$$

$$= z^p + 2(p-\alpha)(p+c)\sum_{n=1}^{\infty}\frac{(-1)^n}{(p+n)(p+n+c)}\,z^{p+n} \qquad (c > -p),$$

one can easily show that $J_{p,c}(f_0(z)) \in \mathcal{P}(p,\mu,\beta)$, but $J_{p,c}(f_0(z)) \notin \mathcal{P}(p,\mu',\beta)$ if $\mu' > \mu$. This completes the proof of Theorem 1.

REMARK I. For $0 \leqq \alpha < p$ and $c > -p$, we have from the right-hand side of (2.6) that $\alpha \leqq \mu < p$, and hence $\mathcal{P}(p,\mu,\beta) \subset \mathcal{P}(p,\alpha,\beta)$ $(\beta \geqq 0)$.

COROLLARY 1. If $f(z) \in \mathcal{P}(p,\alpha,\beta)$, then $F_m(f(z))$ defined by (1.5) belongs to the class $\mathcal{P}(p,\mu_m,\beta)$, where

$$\mu_j = p + 2(p-\mu_{j-1})(p+c_j)\sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c_j} \qquad (j = 1,2,3,\ldots,m)$$

and $\mu_0 = \alpha$. The result is sharp.

Next, we derive

THEOREM 2. If $f(z)$ is in the class $\mathcal{P}(p,\alpha,\beta)$, then $J^{\lambda}_{p,1}(f(z))$ defined by (1.10) belongs to the class $\mathcal{P}(p,\gamma,\beta)$, where

$$\gamma = p + 2(p-\alpha)\sum_{n=1}^{\infty}(-1)^n\left(\frac{p+1}{p+n+1}\right)^{\lambda}.$$

The result is sharp.

PROOF. Making use of (1.7) and (1.8), the definition (1.10) yields

$$(p+1)^{-\beta}D^{\beta}(J^{\lambda}_{p,1}(f(z))) = J^{\lambda}_{p,1}((p+1)^{-\beta}D^{\beta}(f(z)))$$

$$= \frac{(p+1)^{\lambda}}{\Gamma(\lambda)}\int_0^1\left(\log\frac{1}{t}\right)^{\lambda-1}(p+1)^{-\beta}D^{\beta}(f(tz))dt \qquad (\lambda > 0;\ \beta \geqq 0) \tag{2.8}$$

Therefore, setting

$$G(z) = (p+1)^{-\beta}D^{\beta}(J^{\lambda}_{p,1}(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta}D^{\beta}(f(z)), \tag{2.9}$$

we have to show that

$$\mathrm{Re}\left(\frac{G'(z)}{z^{p-1}}\right) > \gamma \qquad (z \in \mathcal{U}) \tag{2.10}$$

whenever $h(z) \in \mathcal{P}(p,\alpha)$. Applying (2.5), we obtain

$$\mathrm{Re}\left(\frac{G'(z)}{z^{p-1}}\right) = \frac{(p+1)^{\lambda}}{\Gamma(\lambda)}\int_0^1\left(\log\frac{1}{t}\right)^{\lambda-1} t^p\,\mathrm{Re}\left(\frac{h'(tz)}{(tz)^{p-1}}\right) dt$$

$$> \frac{(p+1)^{\lambda}}{\Gamma(\lambda)}\int_0^1\left(\log\frac{1}{t}\right)^{\lambda-1} t^p\,\frac{p-(p-2\alpha)t}{1+t}\, dt$$

$$= p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+1}{p+n+1}\right)^{\lambda} . \tag{2.11}$$

To show that the result is sharp, we take the function $f_0(z)$ given by (2.7). Since

$$G_0(z) = (p+1)^{-\beta} D^{\beta}(J_{p,1}^{\lambda}(f_0(z)))$$
$$= z^p + 2(p-\alpha) \sum_{n=1}^{\infty} \left(\frac{p+1}{p+n+1}\right)^{\lambda} \frac{(-1)^n}{p+n} z^{p+n} \tag{2.12}$$

with $0 \leqq \alpha < p$, $\lambda > 0$ and $\beta \geqq 0$, we see that $J_{p,1}^{\lambda}(f_0(z)) \in P(p,\gamma,\beta)$, but $J_{p,1}^{\lambda}(f_0(z)) \notin P(p,\gamma',\beta)$ if $\gamma' > \gamma$. Thus we complete the proof of the theorem.

REMARK 2. For $0 \leqq \alpha < p$ and $\lambda > 0$, we have from the right-hand side of (2.11) that $\alpha \leqq \gamma < p$, and hence $P(p,\gamma,\beta) \subset P(p,\alpha,\beta)$ $(\beta \geqq 0)$.

COROLLARY 2. If $0 \leqq \alpha < p$ and $0 \leqq \beta < \rho$, then $P(p,\alpha,\rho) \subset P(p,\gamma_0,\beta)$, where

$$\gamma_0 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+n}{p+n+1}\right)^{\rho-\beta} .$$

The result is sharp.

PROOF. Setting $\lambda = \rho-\beta > 0$ in Theorem 2, we observe that

$$\begin{aligned} f(z) \in P(p,\alpha,\rho) &\Longrightarrow J_{p,1}^{\rho-\beta}(f(z)) \in P(p,\gamma_0,\rho) \\ &\Longleftrightarrow (p+1)^{-\rho} D^{\rho}(J_{p,1}^{\rho-\beta}(f(z))) \in P(p,\gamma_0) \\ &\Longleftrightarrow (p+1)^{-\beta} D^{\beta}(f(z)) \in P(p,\gamma_0) \\ &\Longleftrightarrow f(z) \in P(p,\gamma_0,\beta). \end{aligned} \tag{2.13}$$

COROLLARY 3. If $0 \leqq \alpha < p$ and $\beta \geqq 0$, then $P(p,\alpha,\beta+1) \subset P(p,\gamma_1,\beta)$, where

$$\gamma_1 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \frac{p+1}{p+n+1} \quad .$$

The result is sharp.

PROOF. Putting $\lambda = 1$ in Theorem 2, we have

$$\begin{aligned} f(z) \in \mathcal{P}(p,\alpha,\beta+1) &\Longrightarrow J^1_{p,1}(f(z)) \in \mathcal{P}(p,\gamma_1,\beta+1) \\ &\Longleftrightarrow (p+1)^{-\beta-1}D^{\beta+1}(J^1_{p,1}(f(z))) \in \mathcal{P}(p,\gamma_1) \\ &\Longleftrightarrow (p+1)^{-\beta}D^{\beta}(f(z)) \in \mathcal{P}(p,\gamma_1) \\ &\Longleftrightarrow f(z) \in \mathcal{P}(p,\gamma_1,\beta). \end{aligned}$$

REMARK 3. We note that the several cases of Theorem 1 and Theorem 2, for the special values of p, α, β, c and λ, will improve some known results.

ACKNOWLEDGMENTS

The present investigation of the second author was supported. in part, by the Japanese Ministry of Education, Science and Culture under a grant-in-aid for general scientific research.

REFERENCES

[1] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429 - 446.

[2] N. E. Cho, On certain classes of p-valently analytic functions, Internat. J. Math. & Math. Sci. 16(1993), 319 - 328.

[3] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl. 38(1972), 746 - 765.

[4] Y. C. Kim, S. H. Lee and H. M. Srivastava, Some properties of convolution operators in the class $P_\alpha(\beta)$, J. Math. Anal. Appl. 187 (1994), 498 - 512.

[5] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104(1962), 532 - 537.

[6] T. Umezawa, Multivalently close-to-convex functions, Proc. Amer. Math. Soc. 8(1957), 869 - 874.

Department of Applied Mathematics
Northwestern Polytechnical University
Xi An, Shaan Xi 710072
People's Republic of China

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan